6.2 Chaos in Logistic Growth Sequences

[...]

The Modeling Methodology. A key theme of this text is the application of mathematics to real world problems through the methods of mathematical modeling. We hope that the reader will gain a realistic understanding of how and why mathematics is useful, and that this understanding will extend beyond the specific details of the various types of models we have considered. In particular, we hope readers will witness the power and applicability of the modeling methodology.

It is appropriate to review this methodology here. We will highlight common aspects of the models we have studied up to this point. Note that these are all nonchaotic models. As we will see, the modeling methodology breaks down for chaotic models.

One key idea we have seen is the distinction between an actual phenomenon or problem context and a mathematical model. The model is a mathematical framework with functions, equations, graphs, and so on. We can apply mathematical procedures (such as algebra) to derive definite conclusions about the model. But we should always remember that these are not directly conclusions about the original problem context.

In almost all cases, models are based on simplifying assumptions. We have seen how such assumptions lead to models of several different types, for example arithmetic growth, geometric growth, etc. We have also seen the use of parameters, and how parameter values can be adjusted to improve the agreement between a model and observed data values. But even when our model agrees very closely with the data, errors remain possible. If our simplifying assumptions are only approximately true, if the observed data can only be measured approximately, then our model can only be expected to approximate the actual context or situation.

Given the approximate nature of a model, we are sometimes more interested in qualitative descriptions of whatever we are modeling than we are in highly accurate predictions of the future. Will the amount of a drug in the blood stream level off? Will a population of fish die out? The models give us qualitative answers to these questions.

The ability to formulate qualitative predictions is closely tied to the appearance of simple numerical and visual patterns in our models. In fact, all of our models have been developed by assuming simple numerical patterns hold in our data, at least approxmately. This is an important feature of our nonchaotic models. We will see that the sequences in chaotic models do not have simple numerical patterns.

There is another important feature of the models we have examined: moderately changing the values of the parameters does not significantly change the qualitative behavior. So, for a geometric growth model, if we have an incorrect value of r, we can still be pretty sure that the long-term behavior will be accelerating growth to ever larger values (for $r > 1$) or a decrease to 0 (for $r < 1$). Similarly, for certain mixed models and logistic growth models, we have found conditions under which the number sequence levels off. For these models, errors in the parameter values need not invalidate the qualitative prediction of leveling off. As long as the errors are within an acceptable range, we know that the number sequence will level off at a predictable value.

Conclusions that remain valid over a range of parameter values are sometimes described as *robust*. For a mixed model, the qualitative conclusion that the model will level off is robust because it doesn't change when the parameter values are slightly modified. This increases our confidence that the same qualitative conclusion will hold for the actual problem context. Even though the mixed model assumptions are only approximately true and the values of the parameters are only approximately correct, because leveling off will occur in a range of model formulations, we expect that it will also occur in the real context. Thus, robustness is an important aspect of using models to make qualitative predictions.