North Polar Ice Cap. In the preceding examples, nearly constant second differences prompted us to adopt quadratic growth models. But data can suggest using a quadratic growth model even when second differences do not appear to be nearly constant. To illustrate this point, we consider a model for the shrinking north polar ice cap, as presented by Witt [**56**].

Our data consist of yearly figures for the extent (in millions of square kilometers) of the north polar ice cap in the month of September each year from 1979 to 2012 [**41**]. See Table 3.8 and Figure 3.7.

Table 3.8. North polar sea ice extent in September, each year from 1979 through 2012. Extent is given in units of millions of square kilometers.

Year	Extent	Year	Extent	Year	Extent	Year	Extent	Year	Extent
1979	7.20	1986	7.54	1993	6.50	2000	6.32	2007	4.30
1980	7.85	1987	7.48	1994	7.18	2001	6.75	2008	4.73
1981	7.25	1988	7.49	1995	6.13	2002	5.96	2009	5.39
1982	7.45	1989	7.04	1996	7.88	2003	6.15	2010	4.93
1983	7.52	1990	6.24	1997	6.74	2004	6.05	2011	4.63
1984	7.17	1991	6.55	1998	6.56	2005	5.57	2012	3.63
1985	6.93	1992	7.55	1999	6.24	2006	5.92		

In Figure 3.7, we have included a straight line representing a possible arithmetic growth model. The data points are scattered fairly widely around the line, in contrast to Figure 3.5 where the data points appear to lie almost exactly on the model curve. Here we recognize that there is a good deal of variability in the extent of sea ice from one year to the next, and our goal is not to predict the exact amount of ice in any particular year. Rather, we would like a description of the underlying trend. These data points do not appear to be scattered at random. There is a visible downward aspect as we proceed from older to more recent values. The line is meant to represent that downward trend, and so is referred to as a *trend line*. In adopting a linear model, we are implicitly assuming that the spread of data points about the trend line in the future will be similar to what we see in the figure. If so, even though we cannot predict the exact amount of ice in any specific year, we *can* predict that the amount of ice will fall within a certain range.

This is illustrated in Figure 3.7 by a shaded area centered on the trend line. For the extant data, nearly all of the points lie in the shaded area, and we expect that to remain true in the future. Applying this in a specific example, we would expect the data point for 2020 to fall within the shaded region. Therefore, based on the figure, we predict that the sea ice extent for 2020 will be between 3.5 and 5.1 million square kilometers.

For this type of analysis, where we look for a trend in a scattered set of points, there are a great many possible lines that might be drawn. On the basis of a casual visual inspection, it would be difficult to single out one specific model. This is where the ability to find the best possible model is useful. The mathematics behind choosing

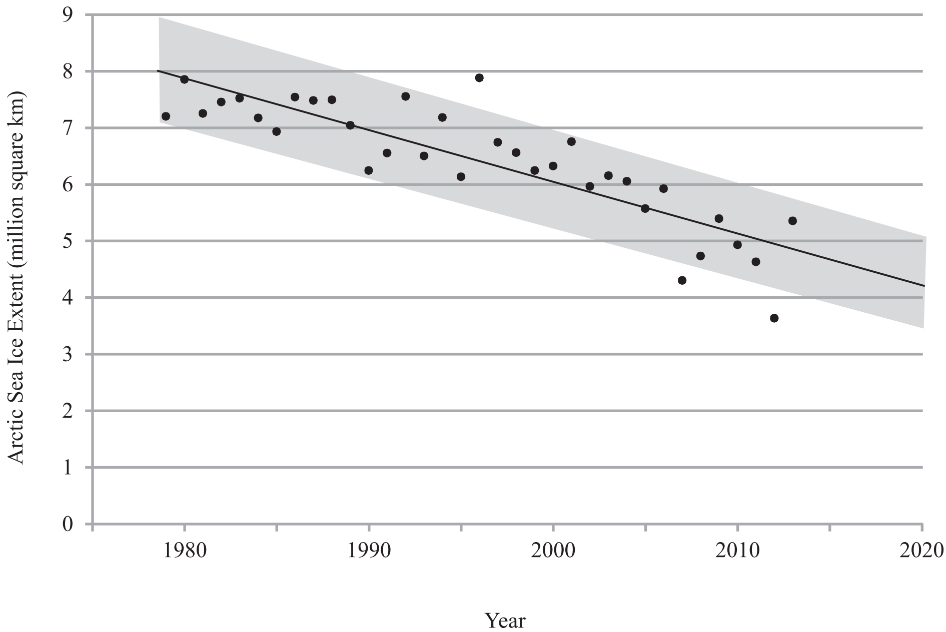

Figure 3.7. September north polar sea ice extent, 1979–2012, shown as dots. The straight line represents the best possible arithmetic growth model. Nearly all of the data points lie in the shaded area, and we suppose that future observations of sea ice extent will continue this pattern.

the best parameters is a bit beyond the scope of the present discussion, but the main idea is roughly to choose a model so that the average vertical distance from the dots to the line is as small as possible. That is the model represented by the line in Figure 3.7.

But why should we choose the best *straight line*? Or equivalently, why should we choose the best arithmetic growth model? Perhaps some other type of model would be better. And indeed, it is not difficult to visualize an arching trend *curve* (instead of a trend line) as in Figure 3.8. This suggests considering a quadratic growth model, even though an analysis of the second differences shows they are nowhere near to being constant.

Just as in the case of a linear model, there are known procedures for defining a quadratic model with the best possible parameters. As before, that means roughly that the average vertical distance from the data points to the graph of the model is as small as possible. However, with a quadratic model, the graph is a parabolic trend curve, not a straight trend line.

The procedures for finding the best quadratic model are pre-programmed in many popular graphing calculators and computer spreadsheet applications. Carrying out this analysis with the sea ice data produces the quadratic model shown in Figure 3.8. As before, the data points are shown as dots, and the quadratic model is shown as a curve. There is also a shaded area indicating how the data points are spread about the curve.

To the eye, the quadratic model does seem to track the cluster of data points more accurately than the linear model. That is not conclusive proof that the quadratic model

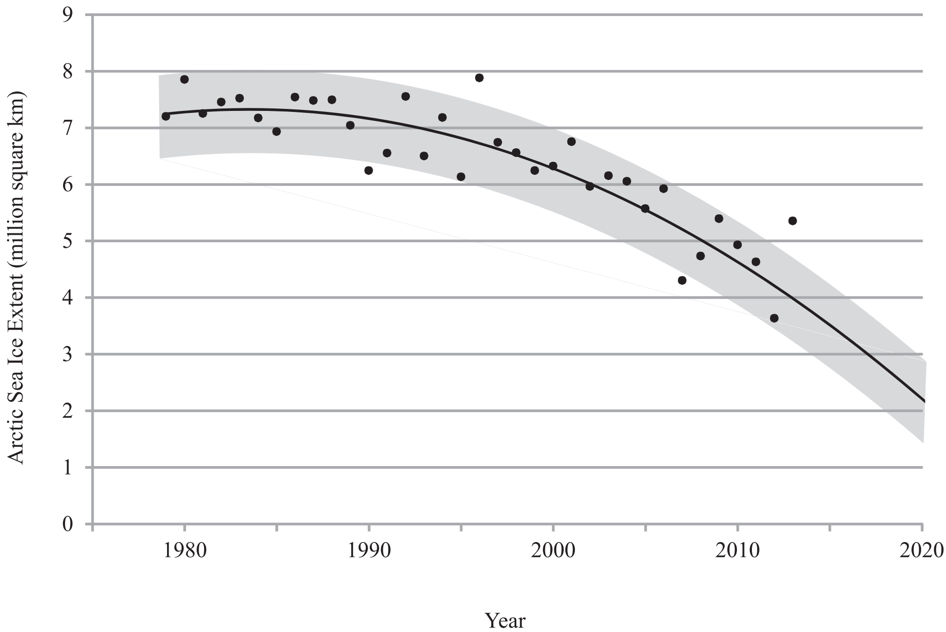

Figure 3.8. The sea ice data with a possible quadratic growth model.

is correct, but without further information, it suggests that the quadratic model is probably more credible than the linear model.

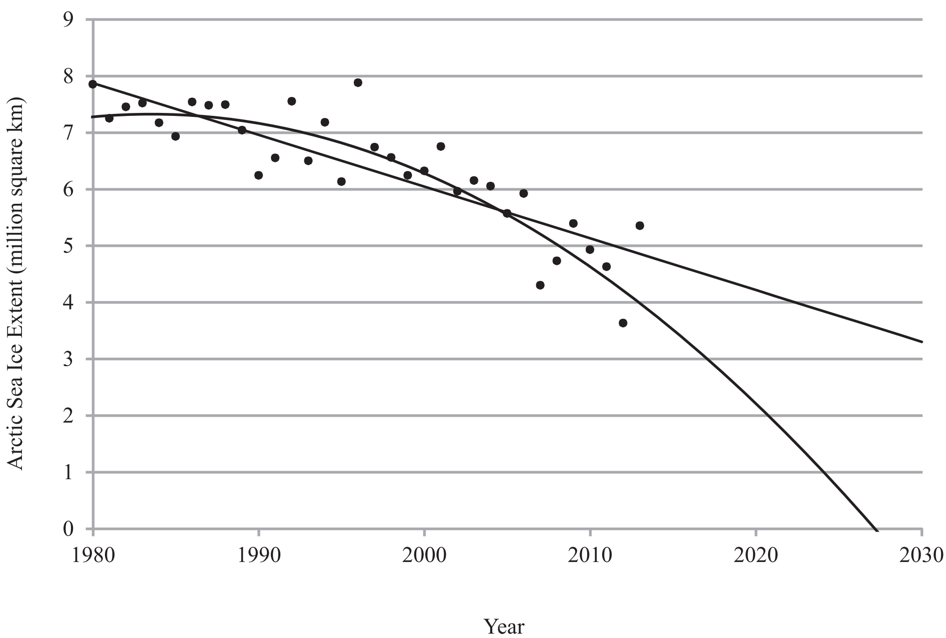

Figure 3.9. Sea ice extent data with linear and quadratic models.

The two models differ greatly in predicting future levels of September sea ice extent. To highlight the difference, we have included both models in a single graph, with the time line extended to 2030. See Figure 3.9. Because the parabolic curve bends away from the line, the further we look into the future, the wider is the separation between the two models. The quadratic model predicts that sea ice could completely disappear in the summer by 2025, whereas the linear model predicts that sea ice extent should still be in a range between 2 and 5 million square kilometers as late as 2030. Remember that we do not predict future levels of sea ice to follow either model curve exactly, but rather to range within an area centered on the curve. Even so, it is interesting to compare where each curve touches the horizontal axis. For the quadratic model that appears to be around 2027; for the linear model the line will not reach the axis until 2066.

As you can see, the two models lead to widely different predictions on the future course of summer sea ice disappearance. Here we are looking at extremely simple models and using relatively unsophisticated methods. But what we have seen hints at the importance and difficulty of validating accurate models for climate change phenomena. At the very least, we can see how a quadratic model might be considered as one possibility, even when the data are far from following the constant second difference rule. And we can see how dramatic an impact the nature of the model can have on what it predicts for the future.

In all of the examples discussed so far, a quadratic model was selected based on a set of data. But quadratic models can be suggested by other considerations. In some applications, the structure of the problem context leads naturally to the selection of a quadratic model. In particular, it may be possible to work out a difference equation based on a logical analysis of the context. If the difference equation has the right algebraic form, we can recognize it as an instance of quadratic growth. We will see several examples of that next.