Entry Level College Math:

Algebra ~~or~~ and Modeling

Dan Kalman

November 2000

Overview

- Algebra can be taught in Modeling courses
- Modeling provides a context for algebra
- Downplay emphasis on unmotivated manipulative skill
- Increase emphasis on genuine applications of algebra
- Kindler, simpler algebraic manipulations

Only teach skills which arise in realistic problems

Plan

- Difference Equation Models
- Sample Progression of Three Topics
  - Geometric Growth
  - Mixed Geometric and Arithmetic Growth
  - Logistic Growth
- Emphasize algebra opportunities

Difference Equations

- Discrete Sequential Data: $a_1, a_2, a_3, \cdots$ and approximating models
- Simple Recursive Patterns
  Examples: each term ...
  - Increases or decreases by fixed amount
    $a_{n+1} = a_n + 50$
  - Increases or decreases by fixed percentage
    $a_{n+1} = 1.2a_n$
- Solutions to difference equations: explicit equation for a_n as a function $f(n)$; extension to continuous models

Analysis Procedures

- Numerical experimentation, exploration
  Algebra: Express general relationships
  Properties of model (function) families

- Fitting a model to actual data by choosing *best* values for parameters
  Algebra: Express problem, parameters

- Direct prediction: evaluate $f(n)$ to predict data value number n

- Inverse Prediction: invert $f(n)$ to predict for which n the data value will reach a specified value
  Algebra: Solve equations

Geometric Growth

- Each term is a fixed multiple of the preceding term; equivalently, each term increases by a constant percentage over the preceding term
- Applications: population growth, compound interest, radioactive decay, drug elimination/metabolization, passive cooling/heating
- Example: population doubles each year (increases by 100%)
  Difference equation $p_{n+1} = 2p_n$
  Solution: $p_n = p_0 2^n$
  Typical questions: What will the population be in year n? When will population reach 60000?

Algebra Skills

- Properties of exponential functions Ab^t
- Graphs
- Significance of parameters A, b
- Solving equations; logarithms

Mixed Growth

- Each term combines a fixed multiple of the preceding term with a fixed increment;
- Applications: amortized loans, installment savings, repeated drug doses, chemical reactions, pollution
- Difference equation: $a_{n+1} = ra_n + d$
- Solutions are shifted exponentials: $Ar^t + C$
- Horizontal asymptote = equilibrium value ($r < 1$)

Example

- Example: Pollution flows out of a lake in proportion to the existing concentration, but flows into the lake at a constant absolute rate

- Difference equation $p_{n+1} = .9p_n + 3$ (one tenth of the pollution flows out, and three more units are added, each unit of time)

- Solution: $$\begin{aligned} p_n &= p_0(.9^n) + 3(1 - .9^n)/(1 - .9) \\ &= (p_0 - 30).9^n + 30 \end{aligned}$$

- Typical questions: What will the pollution load be in year n? When will it reach 100? What will happen in the long term?

Finding the Solution

Numerical pattern:

$$
\begin{aligned}
p_0 &= 20 \\
p_1 &= 20(.9) + 3 \\
p_2 &= 20(.9^2) + 3(.9 + 1) \\
p_3 &= 20(.9^3) + 3(.9^2 + .9 + 1) \\
&\vdots \\
p_8 &= 20(.9^8) + 3(.9^7 + .9^6 + \cdots + .9 + 1) \\
p_n &= 20(.9^n) + 3(.9^{n-1} + .9^{n-2} + \cdots + .9 + 1)
\end{aligned}
$$

Algebraic Simplification

- Solution of difference equation is naturally derived in this form:

$$p_n = 20(.9^n) + 3\left(\frac{.9^n - 1}{.9 - 1}\right)$$

- More convenient equivalent form:

$$p_n = (p_0 - 30).9^n + 30$$

- This shows an important use of algebra: transforming symbolic expressions

- Here, it appears in a natural context generally absent in algebra classes

Equilibrium and Fixed Point

- Traditional Asymptote formulation:

$$\lim_{t \to \infty} f(t) = C$$

- Difference equation formulation (fixed point):

$$p_{n+1} = p_n$$

- Example:

$$\begin{aligned} p_{n+1} &= p_n \\ .9p_n + 3 &= p_n \\ p_n &= 30 \end{aligned}$$

Algebra Skills

- Properties of shifted exponentials $Ab^t + C$
- Graphs, horizontal asymptotes
- Significance of parameters A, b
- Solving equations; logarithms
- Finding fixed points
- Deriving general form of solution
- Transforming Expressions

Logistic Growth

- Modified version of geometric growth. Each term is a multiple of the preceding term, but the multiplier varies linearly with the size of the term
- Example: population p goes up in a year by a factor of $.01(200 - p)$
- Difference equation $p_{n+1} = .01(200 - p_n)p_n$
- No explicit solution, but interesting qualitative behavior: initial growth similar to exponential, but levels off to equilibrium value

General Behavior

- General Difference Equation: $a_{n+1} = m(L - a_n)a_n$
- L is limiting size of population
- Behavior depends on m and L:

$$
\begin{aligned}
0 \le mL < 4 &\Rightarrow a_n \in [0, L] \forall n \\
0 < mL < 1 &\Rightarrow a_n \to 0 \\
1 \le mL < 3 &\Rightarrow a_n \to L - 1/m \\
3 \le mL < 3.5699\ldots &\Rightarrow \text{oscillation} \\
3.5699\ldots \le mL < 4 &\Rightarrow \text{chaos!}
\end{aligned}
$$

Reference:
http://www.mathsoft.com/asolve/constant/fgnbaum/fgnbaum.html

Fixed Points

- Condition: $a_{n+1} = a_n$
- Logistic Growth: $a_{n+1} = [m(L - a_n)]a_n$
- Need $m(L - a_n) = 1$
- Fixed point $= L - 1/m$

Harvesting

- Diff Eqn: $a_{n+1} = m(L - a_n)a_n - h$
- Fixed Point equation

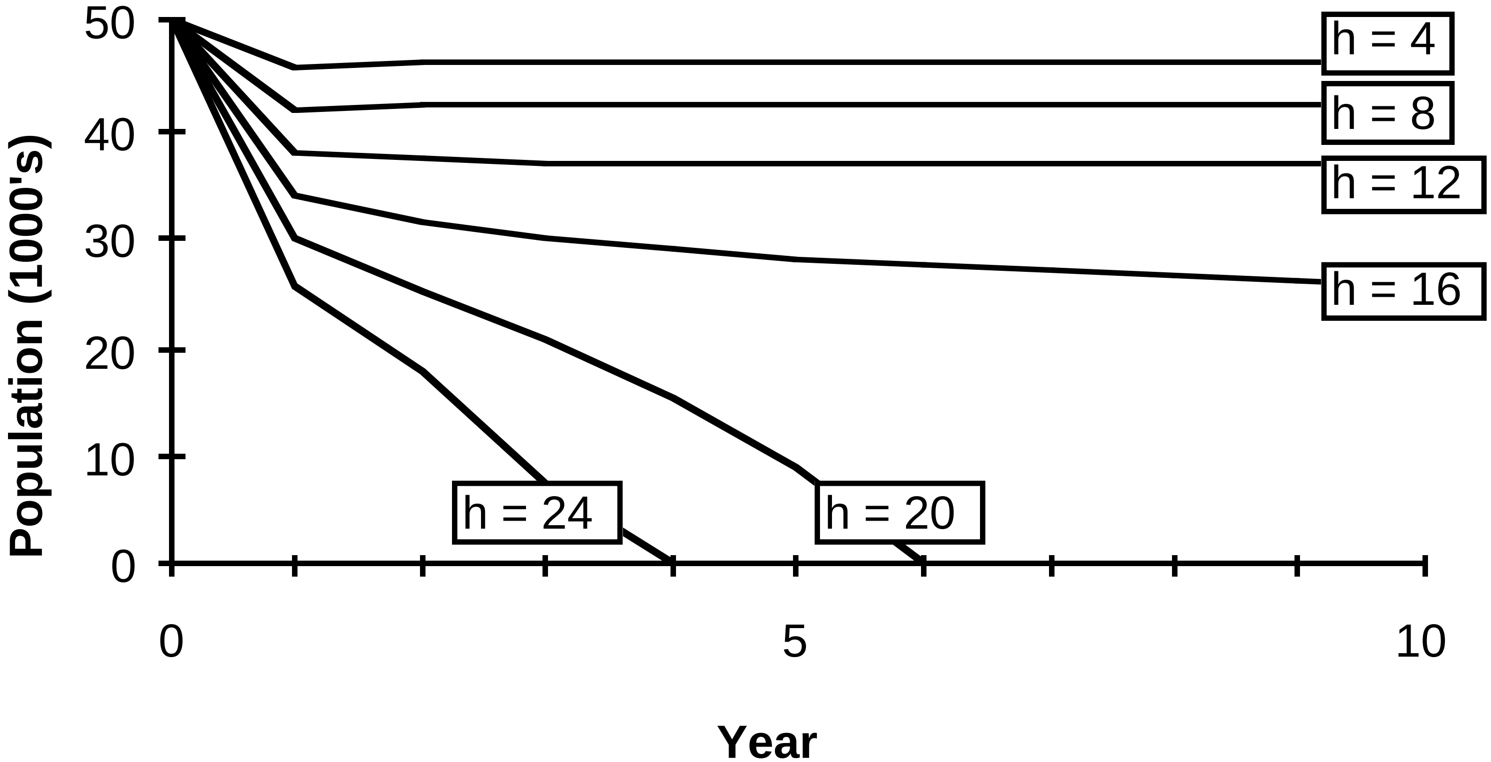
$$\begin{aligned} m(L - a_n)a_n - h &= a_n \\ m(L - x)x - h &= x \\ mx^2 + (1 - mL)x + h &= 0 \end{aligned}$$

- Generally two theoretical fixed points
- Fixed points key to analysis

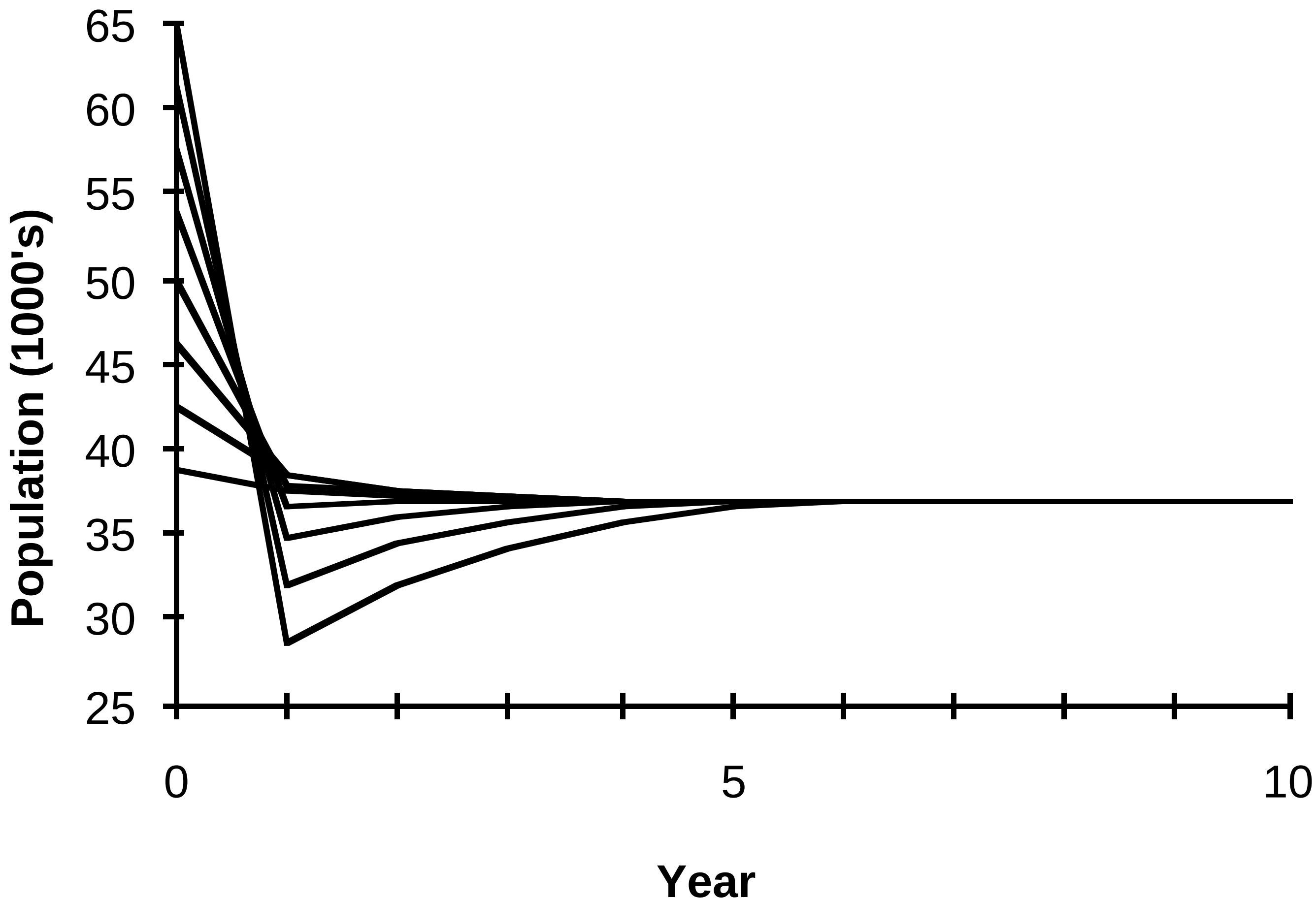
Fish Example

- Diff Eqn: $p_{n+1} = .000025(90,000 - p_n)p_n - h$
- With no harvest, $p_n \to 50,000$
- Graph 1: $p_0 = 50,000$, several h values
- Graph 2: $h = 12,000$, several p_0 values
- Graphs 3,4: $h = 16,000$
- Graphs 5: $h = 15,000$

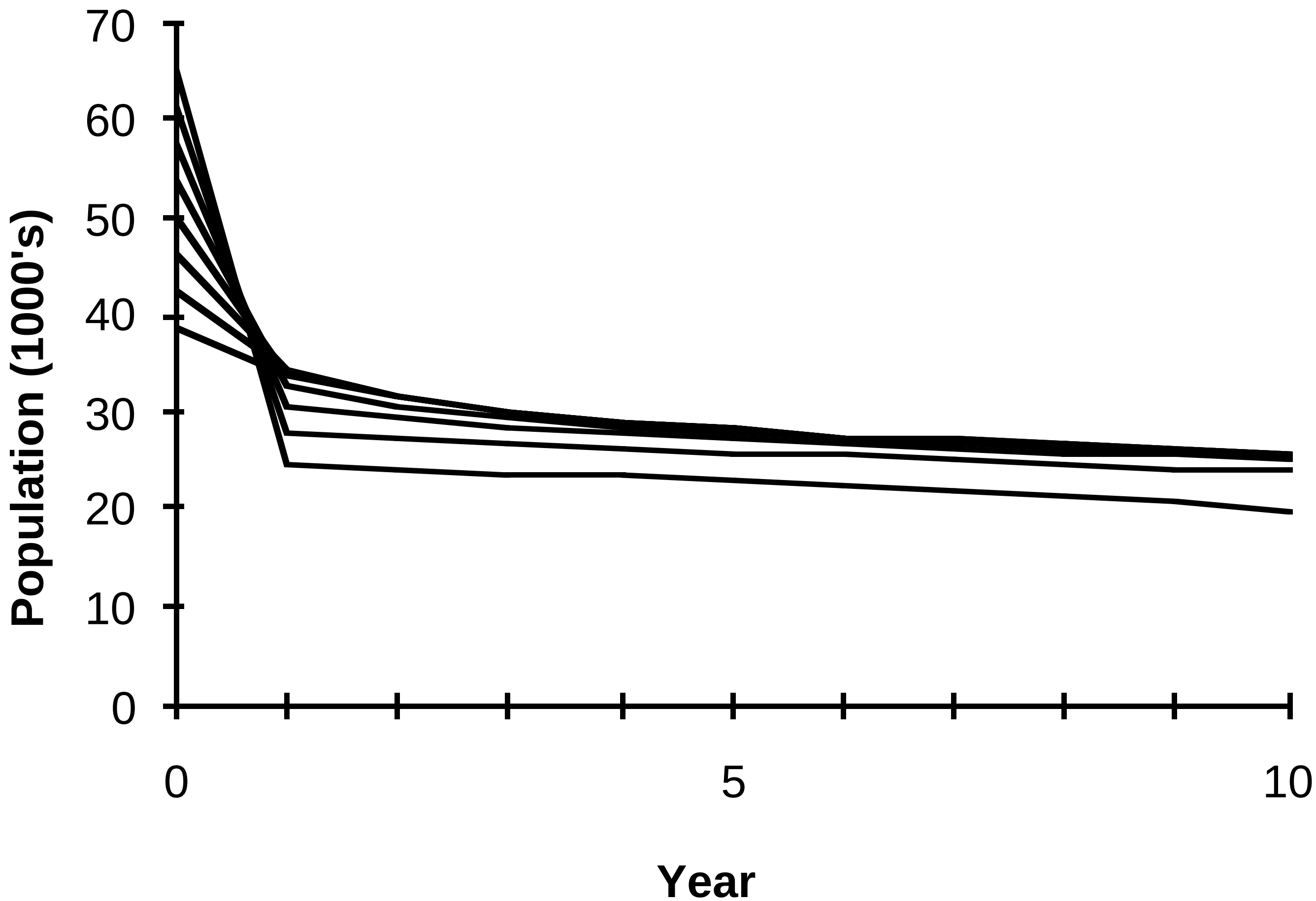
Graph 1: Varying h



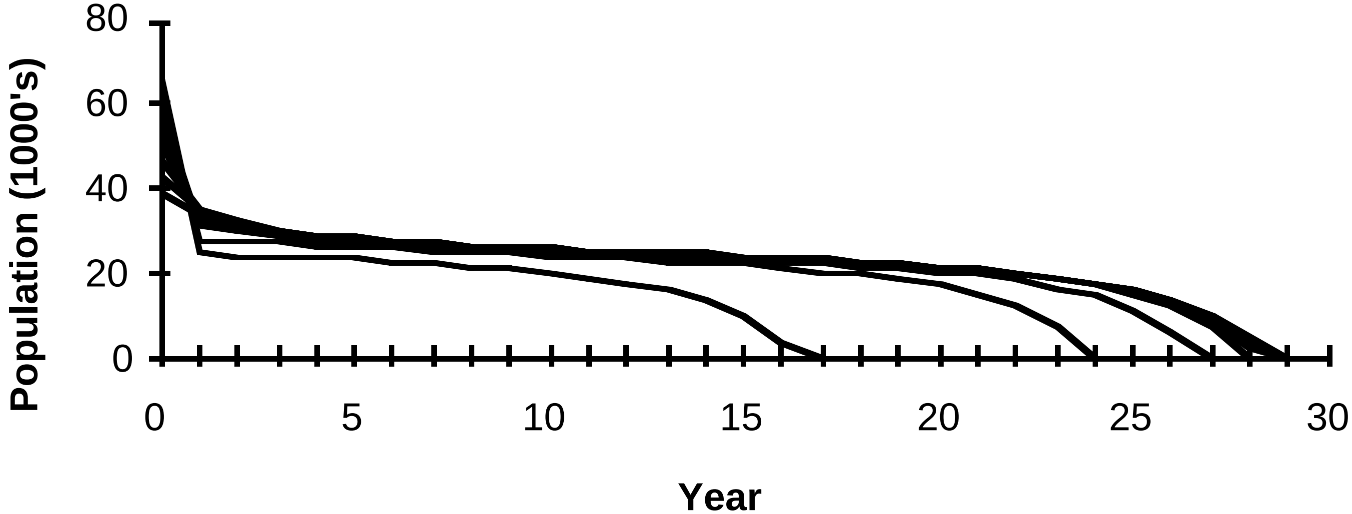
Graph 2: $h = 12,000$, various p_0



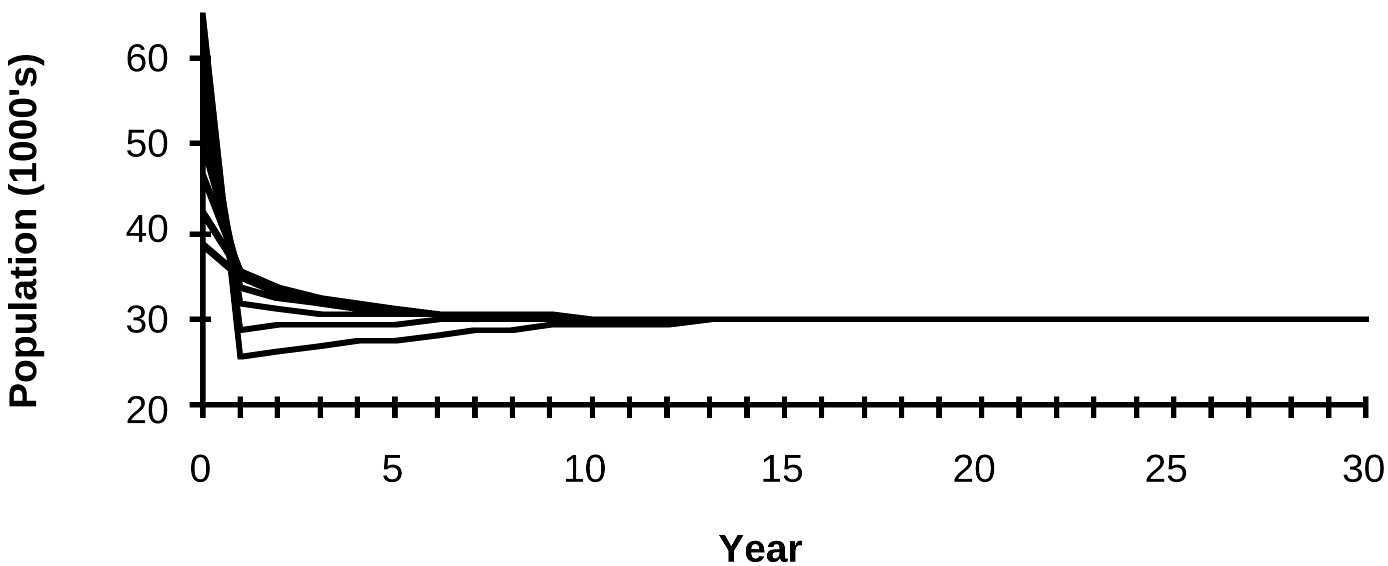
Graph 3: $h = 16,000$



Graph 4: $h = 16,000$

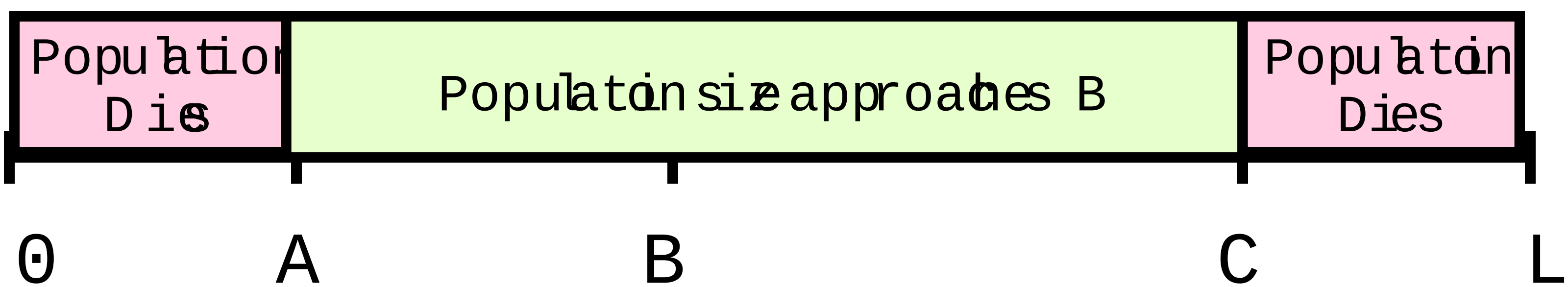


Graph 5: $h = 15,000$



Equilibrium Results

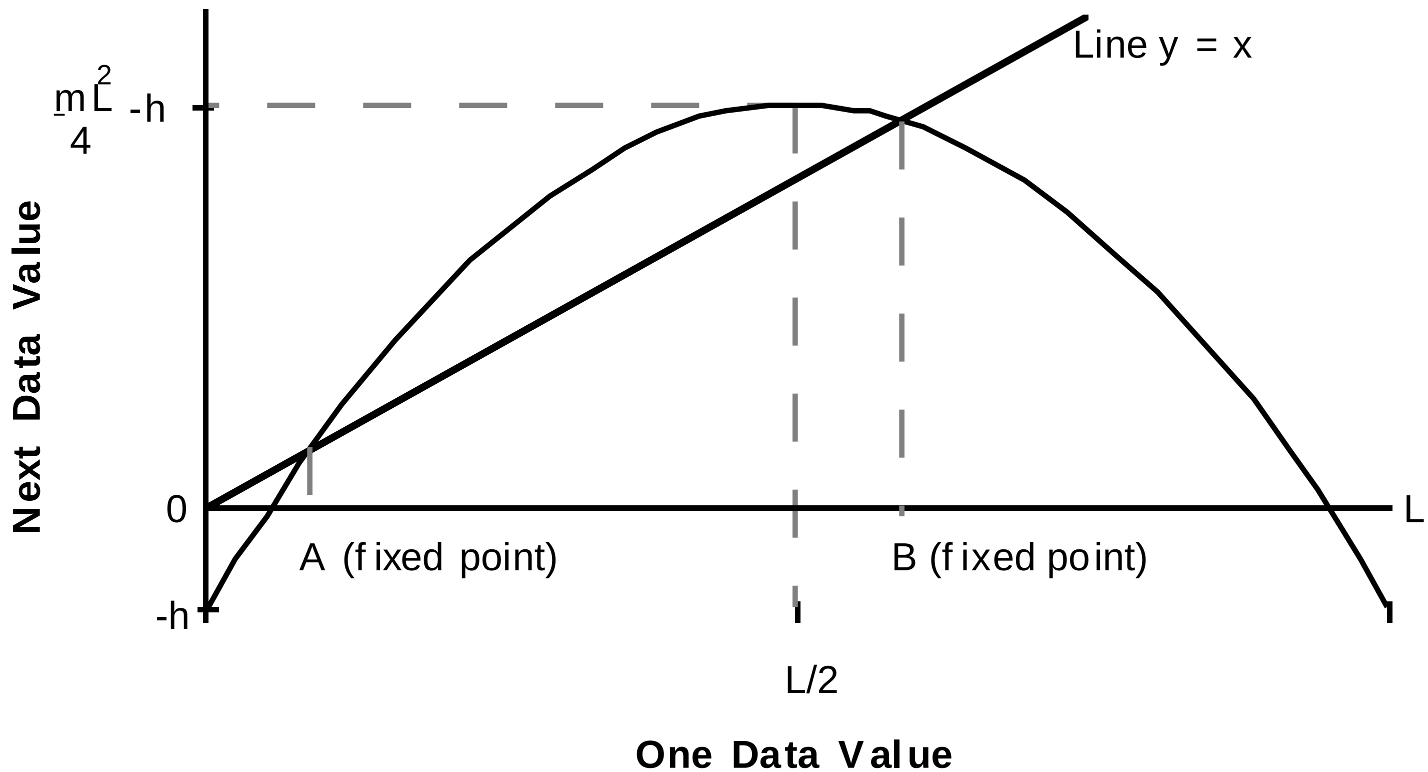
- Two fixed points: $A, B: \; 0 < A < B < L - A \equiv C$
- Partition Interval $[0, L] = [0, A] \cup [A, C] \cup [C, L]$
- $p_0 \in [A, C] \Rightarrow p_n \longrightarrow B$
- $p_0 \in [0, A] \cup [C, L] \Rightarrow p_n \longrightarrow 0$



Derivation

- Key tool: graph of p_{n+1} as a function of p_n
- Graph shows transition from any population size to the succeeding size (see next slide)
- Quadratics: $p_{n+1} = m(L - p_n)p_n - h = -mp_n^2 + mLp_n - h$

Graph: p_{n+1} vs. p_n



Algebra Skills

- Properties of quadratics
- Solving quadratic equations (finding fixed points)
- Inequalities
- Graphical analysis