Elementary Math Models: A hybrid alternative to standard general education math courses

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Materials at emm2e.info/workshopA

Outline

- Motivation: standard course drawbacks
- Goals and Curricular Design Principles
- The Mathematical Story
- Pedagogy
- Classroom Experience

Standard Courses

- College Algebra
  - Needs no description here?
- Liberal Arts Math
  - EG Heart of Mathematics
  - Emphasize positive aspects of math, beyond arithmetic and algebraic technique
- Quantitative Literacy
  - Math skills applicable to everyday life
  - Math methods for *informed participation* in society
- Next: My view of pros and cons

College Algebra

- Pros:
  - Refresher for students who will see quantitative topics in other courses
  - Possible re-entry for students who want/intend to go on in math
- Cons:
  - too abstract; manipulation emphasized too much
  - Ex: $\log_b 84 - \log_b 6 = 1 + \log_b 2$; find *b*
  - Little or no educational significance for many students

Liberal Arts Math

- Pros:
  - Novel topics for most students
  - Broadening view of what math *IS*
  - Significant aspects of aesthetics, psychology, creativity, etc
- Cons:
  - too eclectic
  - May leave students with a distorted view of math
  - Topics that mathies find sexy/cool/awesome/… can fail to land with many students
  - <u>Can be an educational deadend</u>

Quantitative Literacy

- Pros:
  - Topics have obvious significance in everyday life
  - Gen Ed goals RE informed participation in our culture/society
- Cons:
  - Unrealistic to think one course can empower/inspire students to use math tools in life
  - Too eclectic
  - How many people, even those highly math literate, actually use math as shown in QL class?
  - <u>Practitioners vs Consumers of math analyses</u>

Synthesis

- Standard courses have distinct goals:
  a. Polish up math skills for use in other disciplines
  b. Understand and appreciate the significance of mathematics as a tool, as a mode of analysis/thought, as a component of culture
  c. Equip students with a minimal command of math methods for informed participation in society
- All worthy goals, but…
  – Can't do them all justice in one course
  – Some may be infeasible for a single course
- Hybridization: Combine selected aspects of all the above that work together

Course Goals

- Focused immersive experience with mathematical modeling methodology
- Demonstrate the power and utility of this methodology (similar to lib arts math)
- Improve sophistication as a consumer of models based results (not as a practitioner of modeling) (similar to quantitive literacy.)
- Expose students to a coherent mathematical development with significant depth
- Brush up on math skills most likely to be encountered in quantitative aspects of courses outside mathematics (similar to college algebra)

(Some) Design Features

- Evolution of successively more sophisticated growth models, based on plausible and natural hypothesized patterns of growth
- Themes run through the entire course
  a. Developing and refining models
  b. Role and significance of assumptions
  c. Pattern identification, formalization, utilization
  d. Numerical, graphical, algebraic methods
  e. Discrete vs continuous models
  f. Role of parameters; qualitative dynamics
- Emphasize conceptual understanding of how math models contribute to solving problems more than <u>technical mastery of math skills for their own sakes</u>

The Math Story

Discrete Models

- Envision a stream of successive observations as *terms* of a *sequence* $a_1, a_2,$ etc.
- Look for patterns in order to predict future terms
- Recursive vs direct or "functional" patterns
  $a_n \rightarrow a_{n+1}$ $n \rightarrow a_n$
- Difference equation Functional Equation
  $a_{n+1} = f(a_n)$ $a_n = f(n)$
- 1, 4, 9, 16, 25, 36, ...
- Maxim: recursive patterns are easier to see; functional patterns are easier to use.

Model Development and Use

- Observe or hypothesize a recursive pattern
  11, 14, 17, 20, …
  Each term 3 more than preceding term
- Formulate difference equation
  $a_{n+1} = a_n + 3$ or $a_{n+1} = a_n + d$
- Study properties of corresponding family of models (eg. graphs, functional equation)
  Graphs are lines with slope 3 or *d*
  $a_n = 11 + 3n$ or $a_n = a_0 + dn$
- Answer questions about the model.
  Predict the 50th term; When will a value of 1000 or more first be observed?

Progression of Model Types

- Arithmetic Growth; add a constant
  Linear Functions
- Quadratic Growth; added amts grow **arithmetically**
  Quadratic Functions
- Geometric Growth; multiply by a constant
  Exponential Functions
- Mixed Growth; add *and* multiply by constants
  Exponential plus a constant
- Logistic Growth; multiply by factor that depends linearly on current term
  No closed form functional eqn
- Refined Logistic Growth; divide by factor that depends linearly on current term
  Standard Continuous Logistic Functions

Motivating The Progression

- Arithmetic Growth needs no motivation
- Plausible rationales. *Eg.* Geometric Growth
- Structural Analysis. *Eg.* Mixed Models up next
- Revision of earlier models. *Eg.* Logistic and Revised Logistic Models
- Successive models steadily increase in complexity, sophistication
- Continual appearance of important aspects of modeling: assumptions, parameters, fitting models to data; reconsidering assumptions …

Structural Analysis Example

- Pollution level in a lake with clean water inflow
- p_n is a sequence of regularly spaced assays
- Clean inflow leads to a proportional reduction eg. $p_{n+1} = .9p_n$ (10% reduction per time step)
- Plausible to assume pollution continues to be added to the lake at a constant rate.
- Leads to a diff eq. such as $p_{n+1} = .9p_n + 100$
- Pattern analysis reveals:

$$p_n = (.9)^n p_0 + 100\frac{1-(.9)^n}{1-.9}$$

- Simplifies to $p_n = (.9)^n(p_0 - 1000) + 1000$

Example continued

- We found $p_n = (.9)^n(p_0 - 1000) + 1000$
- Can answer questions:
  - What will the pollution level be in 5 years?
  - What will happen in the long run?
  - What if we change assumptions?
- Key features:
  - Simple plausible assumptions
  - Specific quantitative predictions
  - Can compare predictions with real data; refine/revise modeling assumptions
  - Math power: assumptions easily give us difference equation; patterns give us the preferred direct equation; algebra gives us the direct equation in a simpler, more convenient form

Beautiful Evolution of Core Progression

- Arithmetic growth obvious simple 1st guess
- Geometric growth: strong biological rationale
- Logistic growth: Exponential growth clearly unsustainable in long term. Revisit constant multiplier assumption.
- Revised logistic growth: Logistic growth can exhibit chaotic behaviors and negative pop sizes. Revisit assumptions again.
- Bonus: revised logistic growth leads to functional equations comprising an important family of functions.

Side Comment

Revised logistic growth referred to here is a little known bridge between traditional discrete and continuous treatments of logistic growth. See these references for more discussion:

Dan Kalman (2023) *Improved Approaches to Discrete and Continuous Logistic Growth*, PRIMUS, 33:2, 107-122, DOI: 10.1080/10511970.2022.2040664

Dan Kalman (2023) *Verhulst Discrete Logistic Growth*, Mathematics Magazine, 96:3, 244-258, DOI: 10.1080/0025570X.2023.2199676

Technology

- Support experimentation with computational tools tailored to specific growth models
- Suite of excel spreadsheets serving as *apps*
- Also encourage use of graphing calculators – home screen iteration
- Want to make hands-on exploration and pattern generation as easy as possible
- Example: fitting a model to data by trial and error
- Excel *app* demo/activity

Mixed Model Growth

Study various mixed growth models by editing the yellow cells.

difference equation: $a_{n+1} = ra_n + d$

functional equation: $a_n = a_0 r^n + d\left(\frac{1-r^n}{1-r}\right)$

$= d\frac{1}{1-r} + r^n\left(a_0 - \frac{1}{1-r}\right)$

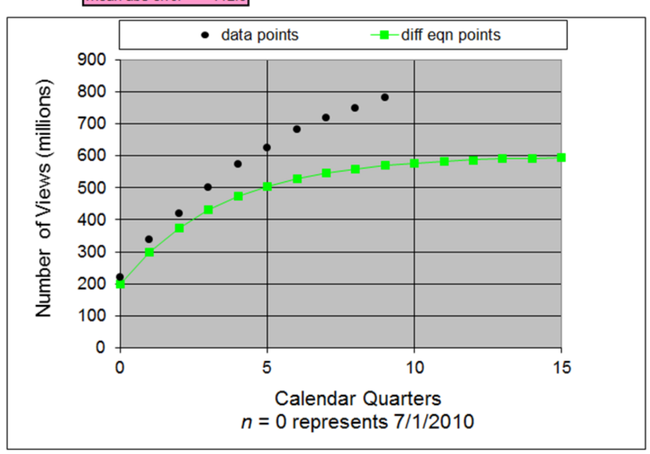
a_0 = 200

r = 0.75

d = 150

mean abs error = 112.6

position number	data values	terms based on diff eqn	absolute errors
0	221.64	200	21.64
1	338.98	300	38.98
2	420.47	375	45.47
3	501.96	431.25	70.71
4	573.66	473.4375	100.2225
5	625.81	505.0781	120.7319
6	681.22	528.8086	152.4114
7	720.34	546.6064	173.7336
8	749.67	559.9548	189.7152
9	782.64	569.9661	212.6739
10		577.4746	
11		583.1059	
12		587.3295	
13		590.4971	
14		592.8728	
15		594.6546	
16		595.991	
17		596.9932	
18		597.7449	
19		598.3087	



Interlude 1:
Fitting Model to Data Activity
(Time Permitting)

SPOILER ALERT

The next slide reveals values of the parameters on the model fitting exercise that are close to optimal, producing a mean absolute error of 5.881. Do not look at it before you have a chance to find your own best fit.

Mixed Model Growth

Study various mixed growth models by editing the yellow cells.

difference equation: $a_{n+1} = ra_n + d$

functional equation: $a_n = a_0 r^n + d\left(\frac{1-r^n}{1-r}\right)$

$= d\frac{1}{1-r} + r^n\left(a_0 - \frac{1}{1-r}\right)$

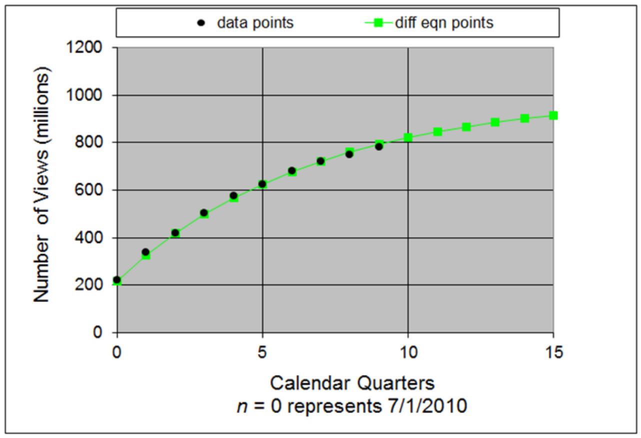
a_0 = 217.3

r = 0.8627

d = 137.2

mean abs error = 5.881

position number	data values	terms based on diff eqn	absolute errors
0	221.64	217.3	4.34
1	338.98	324.6647	14.31529
2	420.47	417.2882	3.181755
3	501.96	497.1946	4.765431
4	573.66	566.1298	7.530245
5	625.81	625.6001	0.20986
6	681.22	676.9052	4.31476
7	720.34	721.1662	0.826151
8	749.67	759.35	9.680038
9	782.64	792.2913	9.651278
10		820.7097	
11		845.2262	
12		866.3767	
13		884.6232	
14		900.3644	
15		913.9444	
16		925.6598	
17		935.7667	
18		944.4859	
19		952.008	



Pedagogy

- Time to mention just a few pedagogical aspects
- Repetition, Repetition, Repetition
- Reading Comprehension, Math Skills & Techniques, Contextualized Problem Solving
- Inductive patterns and functional equations (demonstration)
- Clicker Questions

Pattern to Functional Equation Demo

- Given: $p_0 = 20$; $p_{n+1} = .9p_n + 3$
- Generate terms recursively as follows

  $p_1 = 20 \cdot .9 + 3$

  $p_2 = 20 \cdot .9^2 + 3(.9 + 1)$

  $p_3 = 20 \cdot .9^3 + 3(.9^2 + .9 + 1)$
- Use pattern to predict a later term

  $p_8 = 20 \cdot .9^8 + 3(.9^7 + .9^6 + \cdots + .9^2 + .9 + 1)$
- Predict the general term

  $p_n = 20 \cdot .9^n + 3(.9^{n-1} + .9^{n-2} + \cdots + .9^2 + .9 + 1)$
- Use geometric series identity
- Not a *proof* but students follow this reasoning

Homework Assignment

1. Let a, b, p_0 be unspecified constants, and define

$$p_{n+1} = \frac{p_n}{ap_n + b}$$

for $n \geq 1$. Use the methods of the previous slide/example to find an equation for p_n as a function of n.

2. Make the substitution $q_n = \frac{1}{p_n}$ in the difference equation in part 1, and derive a difference equation for q_n. Find q_n as a function of n as on the previous slide, and then find an equation for p_n as a function of n.

Interlude 2:
Clicker Questions
(Time Permitting)

A group of students is analyzing a technique for purifying a water tank. They repeatedly drain out 90% of the water and replace that with pure water. In their model, n represents the number of times the water is drained and replaced, and p is the amount of contaminants that remain in the tank. According to their calculations

$$p = 400 \cdot 0.9^n + 100\frac{1-0.9^n}{0.1}.$$

In this equation …

A. … p is expressed as a function of n
B. … n is expressed as a function of p
C. … neither variable is expressed as a function of the other

A college student receives an interest free loan of \$10,000 from his grandparents. After graduation, he promises to repay the loan in monthly installments of \$246. After each payment, the amount he still owes decreases. Which of the following best describes the remaining balance after each payment?

a. The remaining balance is an example of exactly arithmetic growth.

b. The remaining balance would be reasonably approximated by an arithmetic growth model.

c. The remaining balance would not be exactly nor even approximately equal to an arithmetic sequence.

S4

Each of the following equations represents a quadratic function. For which one can the y-intercept be found with the least computation?

a. $y = 10x^2 - 12x + 3$

b. $y = -(x + 5)^2 + 4$

c. $y = x(x - 7) + 9x^2$

d. $y = (x - 3)(x + 6)$

S5

A sequence of fractions can be described as follows.

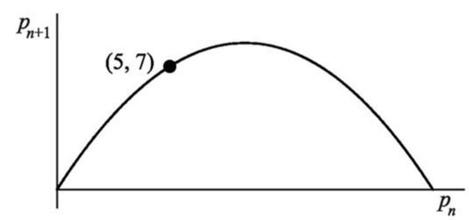
Each numerator is twice the position number and
each denominator is three more than the position number.

Find the fifth and n-th terms in the sequence.

a. $a_5 = \frac{10}{8}$ and $a_n = \frac{2n}{a_n+3}$
b. $a_5 = \frac{10}{8}$ and $a_n = \frac{2n}{n+3}$
c. $a_5 = \frac{10}{13}$ and $a_n = \frac{2n}{a_n+3}$
d. $a_5 = \frac{10}{13}$ and $a_n = \frac{2n}{n+3}$

S6

The graph below shows the next population, p_{n+1}, as a function of this population, p_n, where p_n and p_{n+1} are in units of thousands and n is in units of months. Interpret the point $(5,\ 7)$ shown on the graph.



a. After 5 months, the population is 7 thousand.
b. After 7 months, the population is 5 thousand.
c. If the population is 5 thousand one month then it will be 7 thousand the next month.
d. If the population is 7 thousand one month then it will be 5 thousand the next month.

S7

A scientist has collected desert mammal population data for the past several years. Based on a pattern in the data, he has created a mathematical model in order to predict future populations. Which of the following descriptions of his work is the most likely?

a. He has been able to fully understand and quantify all factors affecting the population.
b. He may have made some simplifying assumptions, but they are not important.
c. He certainly made some simplifying assumptions and they must be taken into account when considering his predictions.

S8

True or False: If a mathematical model matches the known data closely and computations are done correctly, all predictions based on the model will be accurate.

a. True
b. False

S10

Classroom Experience

- Strong buy-in from students
- Those with strongest math prep sometimes object to intellectual demands
- Many positive comments such as

  I am usually awful at math but I really understood this course

  First time I actually enjoyed a math course

  I was dreading math but this turned out to be one of my favorite courses.
- Feasible to cover the entire story (thru revised logistic growth) in 14 week 3 cr. course, but very <u>rushed – no time for term projects</u>

Wrap Up

There is a lot more to say about this curriculum, though possibly not as much more as I have said in print (to paraphrase Paul Halmos).

Links to references are on the webpage, and included on the next slide for future reference.

Thanks for participating.

References

Essays about the EMM curriculum Concept

- "Elementary Math Models: College Algebra Topics and a Liberal Arts Approach," Chapter 32 in Nancy Baxter Hastings et al, eds, A Fresh Start for Collegiate Mathematics: Rethinking the Courses below Calculus, MAA, Washington DC, 2006, pp 304-309. (See page 304)
  (https://maa.org/sites/default/files/pdf/pubs/books/members/NTE69.pdf)
- Entry Level College Mathematics: Algebra or Modeling:
  - AMATYC Review Article Spring 2003 (http://emm2e.info/workshopSE/amatycemm.pdf)
  - Slides based on the article (http://emm2e.info/workshopSE/models & algebra slides.pdf)
- Preface to EMM text (http://emm2e.info/workshopSE/preface from ams page.pdf)

Papers about Refined Logistic Growth
(accessible to MAA members by logging in at MAA.org)

- *Improved Approaches to Discrete and Continuous Logistic Growth*, PRIMUS, 33:2, 107-122, 2023.
  (https://www.tandfonline.com/doi/full/10.1080/10511970.2022.2040664)
- *Verhulst Discrete Logistic Growth*, Mathematics Magazine, 96:3, 244-258, 2023.
  (https://www.tandfonline.com/doi/full/10.1080/0025570X.2023.2199676)