Preface to Second Edition

Elementary Mathematical Models (EMM) is for students at the level of college algebra or precalculus. But it does not fit the mold of these or the other courses one typically sees at this level, including liberal arts mathematics and quantitative literacy courses. Because EMM has instructional goals in common with all of the above mentioned courses, it may be considered a hybrid of sorts. And because it is distinct from the standard courses, potential readers and instructors are entitled to a detailed description of EMM's underlying philosophy and goals, as well as the topics covered. The first part of this preface is provided for that purpose. There will follow a few remarks about instructional methods, experiences from EMM teachers, and new features of the second edition.

Students Served. EMM is intended to serve the same students as traditional college algebra, liberal arts mathematics, and quantitative literacy courses. These are students who have studied at least a year of algebra in high school, but who are not headed for calculus. They may need to take a mathematics course for a general education requirement. They may also need familiarity with the main ideas of college algebra for quantitative general education courses in areas outside of mathematics, such as science, economics, and business. For most of these students this will be the only mathematics course completed in college.

EMM has also been used as a mathematics content course in teacher education programs. In one instance it was part of a master's program in Mathematics Education for middle and high school teachers. The instructor found that the material was accessible to participants with an elementary education background, while still novel and engaging enough to interest those with an undergraduate mathematics major. Similarly, at another institution, EMM was part of a summer program for in-service middle school teachers seeking to add a mathematics endorsement to their teaching credentials. As an example at the undergraduate level, one college offered EMM as one of six options fulfilling a general education requirement. There, majors in the Education program were advised to take EMM because it covers topics closely related to middle and secondary school curricula, while also challenging students to deepen and extend their understanding of those topics.

Development Philosophy. The development of EMM reflects two important concerns. First, we want the course to possess an internal integrity. The aim is to make the material intrinsically interesting and worthwhile, to present a coherent story throughout the semester, and to convey something of the utility, power, and method of mathematics. Among other things, this demands genuine and understandable applications.

In formulating these goals for the first edition, Kalman was strongly influenced by the innovative statistics text by Freedman, Pisani, and Purves.[1] The authors' account[2] of the development of this book is a model of clarity and insight. It describes a succession of attempts to refine an introductory statistics course. Ultimately, the authors were forced to focus on a single fundamental question: What are the main ideas the field of statistics has to offer the intelligent outsider? Their answer to this question became the foundation for a new course in statistics. In the same way, EMM tries to capture some of the main ideas that arise in the applications of mathematics, and to present these ideas in an interesting and intelligible context.

Course development was also guided by a second fundamental concern, to emphasize the topics that students are most likely to meet in mathematical applications in other disciplines. We are thinking here of the most elementary applications, formulated in terms of arithmetic and simple algebraic operations: linear, quadratic, polynomial, and rational functions; square roots; and exponential and logarithmic functions. These functions are the building blocks for the simple models that appear in first courses in the physical, life, and social sciences. Many students in our course have learned about some or all of these functions in prior classes. The challenge is to make the material fresh and interesting for these students and, at the same time, accessible as a first exposure. EMM tries to do this by allowing each mathematical topic to appear naturally in the context of an application or simple recursive growth model.

Throughout EMM applications are analyzed using a common methodology involving difference equations. This point will be discussed in greater detail presently. It suffices here to indicate that difference equation methods are applicable to a number of problem contexts that (we hope) have obvious significance and relevance. The mathematical topics we cover arise as a consequence of studying difference equation models. The algebraic emphasis is restricted to what is really required for working with these simple models.

This is one distinction between EMM and standard college algebra or precalculus courses. Such courses typically aim to train students in techniques of algebraic manipulation, covered abstractly and in encyclopedic breadth. Students are asked to master algebraic operations that they will seldom see again, if ever, even in more advanced mathematics courses: the simplification of complicated rational functions, compound radical expressions with several different radices, and logarithmic equations with variable bases, to name a few. EMM de-emphasizes algebra facility as an end itself.

That is not to say that we have eliminated all algebra. Far from it. Algebra is an important and powerful tool, and students should appreciate and remember that fact. But we want the students to make that observation themselves, from seeing algebra in action. The algebra that does appear in the book is presented in a way that makes it clear *why* algebra is needed, and what it contributes to formulating and analyzing models.

EMM is a hybrid of liberal arts mathematics courses and traditional college algebra courses. Many of the goals of instruction are like those of a liberal arts mathematics course. We wish students to experience and appreciate the power of mathematical methods, the role of aesthetics. We have attempted to present the material in a way that allows students to fully understand the logic behind our methods and how we

[1] David Freedman, Robert Pisani, and Roger Purves. *Statistics.* W. W. Norton, New York, 1978.

[2] See the first 6 pages of the introduction to the instructors manual for the text.

know our conclusions are valid. We also hope they will have a positive experience studying this material, and recognize the applicability of mathematics to topics with evident relevance to students' lives. At the same time, by emphasizing modeling and elementary functions, EMM prepares students for the mathematical topics that will arise in their other courses.

EMM also shares goals with courses in quantitative literacy. We hope that students will develop and retain an understanding of the modeling methodology that underlies so many real applications of mathematics. However, whereas many quantitative literacy courses emphasize topics that students might apply in their own daily lives, EMM is more concerned with providing a basis for understanding how mathematical methods lead to conclusions about climate, public health, ecology, and other significant concerns. We do not necessarily hope students will be practitioners of applied mathematics. We do want them to be informed consumers of the conclusions of applied mathematics. As one aspect of this, we have a consistent emphasis on identifying and critiquing assumptions of models, and on the limitations these assumptions impose on results derived from models.

Based on the concerns discussed above, the development of EMM was guided by the following principles:

- Introduce new mathematical operations in the context of a believable problem or plausible growth assumption;
- Weave all of the topics into an integrated whole;
- Provide as a theme a methodology that can be used in many problem contexts;
- Emphasize conceptual understanding of how the mathematics contributes to solving problems over technical mastery of each mathematical topic for its own sake.

The second and third principles are reflected in the choice of topics for the course. A brief discussion of the organization of topics will be presented next.

Course Outline. In overview, EMM concerns discrete and continuous models of growth. Each application starts with a discrete model built around a sequence $\{a_n\}$. There is also a simple hypothesis describing the way successive terms depend on preceding terms. One example of such a hypothesis is that each new term a_{n+1} can be obtained by adding a fixed constant to the preceding term, a_n. The algebraic expression of this hypothesis,

$$a_{n+1} = a_n + \text{a constant},$$

is a *difference equation*. Throughout the course we consider a succession of simple hypotheses: the terms of the sequence increase by a constant amount; the terms increase by varying amounts which themselves increase by a constant amount; the terms increase by a constant percentage; and so on. Each of these hypotheses finds expression as a particular kind of difference equation. Each new class of difference equation is studied using a common methodology, as follows.

- Formulate a family of difference equations;
- Develop *solutions* to the difference equations;
- Study the behavior of the solutions.

While the models are initially formulated in terms of discrete variables, the solutions of the difference equations can generally be interpreted in the context of continuous variables.

The course starts with a discussion of sequences and number patterns, and introduces difference equations and appropriate algebraic formalisms for describing sequences. Next we introduce arithmetic growth, before proceeding to build up successively more complicated models: quadratic growth, geometric growth, mixtures of arithmetic and geometric growth, and finally logistic growth. In the penultimate section, EMM discusses the chaotic behavior that can occur in logistic models. Finally, we consider whether chaos might be an artifact of the assumptions of our logistic growth model. By modifying one assumption in a natural way, we derive a revised version of logistic growth that avoids chaos and leads to the standard family of continous logistic growth functions.

Interspersed among the discussions of the various growth models are units on the families of functions that appear as solutions to difference equations. Thus, the study of arithmetic growth leads into a unit on linear functions; quadratic growth models provide a setting for studying the properties of quadratic functions, and so on.

Unifying Themes. Throughout the course, several themes are touched on repeatedly. Obviously, the formulation of a model in terms of a difference equation, and the development of a solution to that difference equation has a constant presence. Another theme pertains to the distinct roles played by difference equations and their solutions in studying discrete models. It is repeatedly stressed that simple recursive patterns, expressed as difference equations, are a natural way to describe a sequence $\{a_n\}$, while expressing a_n as a function of n is a powerful tool for analyzing the sequence. The use of systematic investigation and patterns to discover the solutions of difference equations is a third common theme. In particular, students repeatedly see the formulation and solution of parametric families of difference equations by such methods. Numerical and graphical methods are used systematically in all the topics.

The progression of topics demonstrates another theme, the incremental nature of modeling. We repeatedly see earlier models refined or modified to obtain later, more realistic models. Students are encouraged to think critically about the assumptions incorporated in our models, and how inevitable simplifications can dilute the accuracy of conclusions derived from models. They see a dramatic evolution of simple growth models, from arithmetic, to geometric, to discrete logistic, and ultimately, to a refined version of discrete logistic growth.

Finally, the most fundamental theme of EMM is the applicability of the mathematical topics. Students recognize this from the start, because the entire course evolves out of the investigation of real problems. In this second edition we have increased the emphasis on modeling situations using real data, about topics that students recognize as important. And by the end of the course, they are studying sophisticated models with unexpected and nontrivial behaviors. The power of algebra to answer important questions about these models is always on display.

Syllabus. EMM was designed as a three-credit-hour course extending over a semester with 14 weeks of instruction. In several semesters of class testing, meeting 75 minutes twice per week, Kalman found it possible to cover almost the entire text, omitting or mentioning only briefly sections 3.4 and 4.5. An advantage of this approach is that students see the full range of models, with the final two topics providing a nice capstone for the course.

Some instructors may prefer to cover less material in greater depth or at a slower pace. In particular, the inclusion of student projects would nicely complement the modeling emphasis of the course, but would be difficult to include if the entire text is covered. Omitting all or part of Chapter 6 would be one way to provide for additional instructional time for the remaining topics.

For use in a ten-week quarter, the first four chapters might constitute an appropriate syllabus. Or, in two quarters, the entire text might be covered with additional instructional time devoted to students projects.

The instructor's guide provides more detailed suggestions related to specific topics or sections of the text.

Teaching Methods. The methods used to teach EMM are independent of the course content. We believe the material could be successfully presented in a very traditional format. Our test classes have used a combination of teaching methods. The students listen to some lectures, and complete fairly traditional homework assignments. But they also work in groups and do reading and writing assignments. With the first edition, they also developed models based on data they found in magazines or newspapers, and investigated aspects of models in a computer laboratory. We have tried to develop the material for EMM in a way that lends itself to a variety of approaches.

Nevertheless, we feel that the material works best when there is a strong emphasis on reading and writing. There are reading comprehension questions throughout the text, and we work hard at the beginning of the course to get the students to actually *read* their math books. This is a new experience for many students, and it may require some coaching. As teachers, we are well aware that reading a math book with comprehension requires different strategies and skills than for other kinds of literature. Students generally do not know that this is the case.

Websites. Additonal resources are available at two websites. At a public site of ancillary resources [**22**] students and teachers can access a suite of Excel spreadsheets and a technology guide. Through the AMS, teachers can access an instructor's guide, a collection of *clicker questions*,[3] and a compilation of all exercises from the text with selected answers and solutions.

Technology. EMM lends itself to numerical exploration and experimentation. We highly recommend the use of calculators or computers for this purpose. This makes numerical and graphical analysis quicker and easier than paper and pencil methods, and produces results that are more accurate, both numerically and visually.

In our test classes, we have used a combination of graphing calculators and Excel spreadsheets. We have developed a suite of spreadsheets that allow students to enter data and parameters for various growth models, and instantly obtain graphs and tabular results. More details about spreadsheets, as well as calculators and other technology options, appear in the technology guide. The guide and the collection of spreadsheets are available online [**22**].

[3]These are multiple choice questions for in-class use as part of a peer-learning activity with a classroom response system (aka *clickers*). More detailed information is provided with the collection of questions.

Classroom Experience. Kalman has frequently taught EMM at American University for over 20 years. From 2015 to 2018 he used drafts of the new edition as they were produced. The first edition of EMM has also been used at a number of other institutions, including Columbus (Georgia) State University, Evergreen State College, Hollins University, Hood College, and Messiah College, among others. Anecdotal evidence from American University and the other institutions mentioned shows that the student reaction has been generally favorable. The students seem to appreciate the fact that the applications are so tightly integrated into the development of the ideas. They also have generally found the course to be demanding. But on the whole, they find the material within reach.

At American University the students have diverse backgrounds. Those whose preparation is weak find the text quite challenging. Better prepared students sometimes complain that the prose is repetitive or long-winded. In most cases, we have erred on the side of too little reliance on the power of algebra to summarize and justify conclusions, and students with a superior understanding of symbolic methods might sometimes perceive the exposition as inefficient. In spite of this, or maybe because of it, our students have generally liked the text.

For almost all of our students, EMM (or some alternative mathematics course) is a requirement. Some of these students are unmotivated or do not engage with the material. But even students who are engaged sometimes find the material quite difficult. In a class of thirty, it is not unusal to find one or two who really seem to make an honest effort but struggle to succeed in the course. Overall, though, we have found students to be interested in the material and to complete it successfully.

This conclusion is supported by student remarks on teaching evaluations. Each semester, there are a few students (perhaps three or four in a class of thirty) who make comments like these:

> *Best math course I ever took; I am usually awful at math but I really understood this course; first time I actually enjoyed a math course; I was dreading math but this turned out to be one of my favorite courses.*

There also are always a few students who object to the emphasis on writing and *thinking*. It is rare for a student to find the material so easy that it offers no intellectual challenge. On the other hand, the stronger students would find that a traditional college algebra course can be completed in a much more mechanical way and with much less in the way of conceptual demands. Perhaps these are the same students who complain: *Too much writing. I never had to write essays in a math class before.*

Changes in the Second Edition. This book is not so much a new edition of *Elementary Mathematical Models* as it is a *reboot*. We have retained the original aims, philosophy, and design principles as in the first edition. But almost the entire text has been rewritten, with updated examples and a greater emphasis on genuine applications using real data. Based on the experience of teachers over the past 20 years, we have reorganized the presentation, dividing chapters into separate sections, each with its own set of exercises. Some of the material from the first edition has been omitted, including a derivation of linear regression formulae, aspects of logarithms and logarithmic scales, and a unit on polynomials and rational functions. One new unit has been included, deriving the family of continuous logistic growth functions from a modified discrete logistic growth difference equation.

We have made an effort to include a richer collection of exercises in the second edition. Answers are provided for about half of the exercises (marked with an @ symbol), including in many cases a detailed solution. There are also separate sections of *Digging Deeper* exercises that are more challenging or sophisticated, or that go into greater depth.

With this edition we are also providing several ancillary resources for teachers and students. These include the previously mentioned teacher's guide, extended solutions manual, and a *clicker question* collection for teachers, as well as the technology guide and Excel spreadsheet collection for both teachers and students.

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Several colleagues have contributed to this new edition. Jon Scott and Angela Hare were directly involved in parts of the project, reviewing and critiquing material, topics, and exercises. Adam Mills-Campisi, a former student of Dan Kalman, wrote exercises for the Supplementary Problem Collection mentioned above, including some that were incorporated in the new edition. Other colleagues contributed by sharing their experiences with EMM over the years, including Julie M. Clark, Ladnor Geissinger, Kitt Lumley, Elizabeth Mayfield, and Neal Nelson.

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